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THE SET THEORETIC AMBIT OF ARROW'S THEOREM

ABSTRACT. Set theoretic formulation of Arrow's theorem, viewed in light of a taxonomy of transitive relations, serves to unmask the theorem's understated generality. Under the impress of the independence of irrelevant alternatives, the antipode of *ceteris paribus* reasoning, a purported compiler function either breaches some other rationality premise or produces the *effet Condorcet*. Types of cycles, each the seeming handiwork of a virtual voter disdaining transitivity, are rigorously defined. Arrow's theorem erects a dilemma between cyclic indecision and dictatorship. Maneuvers responsive thereto are explicable in set theoretic terms. None of these gambits rival in simplicity the unassisted escape of strict linear orderings, which, by virtue of the Arrow–Sen reflexivity premise, are not captured by the theorem. Yet these are the relations among whose n-tuples the *effet Condorcet* is most frequent. A generalization and stronger theorem encompasses these and all other linear orderings and total tierings. Revisions to the Arrow–Sen definitions of 'choice set' and 'rationalization' similarly enable one to generalize Sen's demonstration that some rational choice function always exists. Similarly may one generalize Debreu's theorems establishing conditions under which a binary relation may be represented by a continuous real-valued order homomorphism.

Arrow has recounted that an early interest in mathematical logic, piqued by the work of Russell and a course from Tarksi, formed the seedbed for his celebrated impossibility theorem on the 'rational' aggregation of certain binary relations.[1] Arrow's departure from the nomenclature of set theory to some extent clouds the set theoretic compass of the theorem. This point has itself been eclipsed in a rich discussion of related impossibility results and of consequences for economic decisionmaking. The following treatment first casts the theorem and the occurrence of cycles in precise set theoretical terms. It then traces in the same terms responses to the dilemma that the theorem poses. The fruit is the discovery that the theorem is at once uneconomical and unnecessarily restricted in a respect whose cure furnishes a stronger result and generalization. The same is also true of Sen's companion results on the inferability of certain choice functions. These generalizations may seem within economics to enlarge scope only by including relations infrequently exhibited by consumers. Their general theoretical significance lies in the sweep of conclusions across all manner of transitive relations that position each member of a set with respect to each other.

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To aid clarity, and against a history of inconsistent terminology concerning orderings and other binary relations, a taxonomy of transitive binary relations appears in Figure 1. The following definitions apply. A relation R is said to be *transitive* if and only if $xRy \cdot yRz \rightarrow xRz$. R is *symmetric* if and only if $xRy \rightarrow yRx$. Where *antisymmetry* is expressed by the conditional $xRy \cdot yRx \rightarrow x = y$, an *asymmetric* R is an antisymmetric R for which the antecedent is always false. A *nonsymmetric* R is neither symmetric nor antisymmetric, which is to say that $xRy \cdot yRx$ obtains for some but not all distinct x and y (as in 'is a brother of'). R is *connected* on a set S if for all distinct $x, y \in S$, $xRy \vee yRx$, which is to say that R compares every element to every other. R is *reflexive* if and only if xRx for all $x \in S$, *irreflexive* if and only if xRx for no x, and *nonreflexive* if and only if xRx for some but not all x. An asymmetric relation is always irreflexive and a transitive irreflexive relation is always asymmetric. To name a transitive nonsymmetric relation, important in the discussion to follow, we introduce the term *tiering*. This term suggests a structure in which many elements of a set are positioned at the same level.

A connected (or *linear*) ordering may be represented by a line, a nonconnected (or *partial*) ordering by a forest of trees, a connected (or *total*) tiering by a staircase, and a nonconnected (or *partial*) tiering by a collection of staircases. A *preordering* (or *quasiordering*) is a transitive and reflexive relation (of which there appear four in Figure 1). Some orderings (the irreflexive and nonreflexive) are not preorderings.

Alternatively to Figure 1, a tree of transitive relations might branch first into irreflexive, reflexive, and nonreflexive relations. Such a tree would mistakenly represent that asymmetric orderings are not antisymmetric and would also fail to group nonreflexive linear orderings with other linear orderings. Yet another tree might branch first into connected and not connected relations. That would not group together those relations, namely, orderings, in which no two distinct relata occupy the same level. Figure 1 is predicated upon the most economical definition of an ordering, to wit, a transitive antisymmetric relation.[2] Speaking of economy, a strict linear ordering may be identified as that transitive connected relation on S of least cardinality. If to a strict linear ordering one adds $\langle x, x \rangle$ for all x, one obtains a weak linear ordering. By adding to the latter $\langle y, x \rangle$ for some but not all $\langle x, y \rangle$ when $x \neq y$, one produces a weak total tiering. Weak total tierings are the transitive connected relations on S of greatest cardinality.

Relations identified in Arrow's theorem are marked '$\rightarrow$' in Figure 1, while '$\nearrow$' denotes additional relations embraced by numbered clauses of the theorem in part IV.

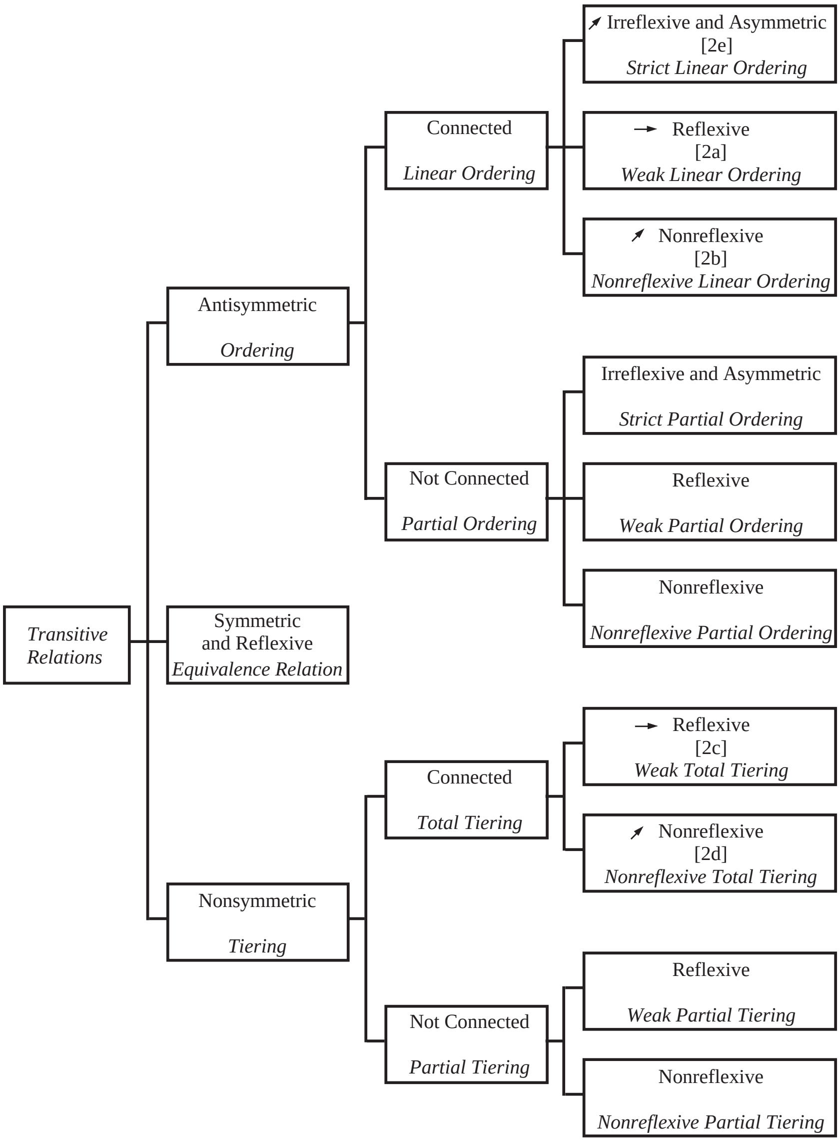

Figure 1. SUBSETS OF THE SET OF TRANSITIVE BINARY RELATIONS.

I

STATEMENT AND PROOF OF THE THEOREM

In respect of a set S, Arrow specified a universe of binary relations R satisfying two axioms:

$$\text{I.} \quad \forall x, y \in S(xRy \vee yRx).$$

$$\text{II.} \quad R \text{ is transitive.}$$

Arrow explained that axiom I demands not only connectivity but reflexivity (the latter from the case $x = y$, as it is not stipulated that x and y be distinct) and allowed that he was using 'slightly inaccurate terminology' when he called this axiom 'connectivity' for short.[3] Given any transitive R, economists customarily define by

$$xIy \Leftrightarrow [xRy \cdot yRx]$$

a subrelation I of 'indifference,' and by

$$xPy \Leftrightarrow [xRy \cdot \sim(yRx)]$$

a subrelation P of 'preference.' Thus $R = P \cup I$. Because P is asymmetric and I symmetric, R is partitioned into an ordering and an equivalence relation. (Arrow defined P by $xPy \Leftrightarrow \sim[yRx]$. Though this coincides with our definition above for reflexive R, in the irreflexive case of interest below, $[xRy \cdot \sim(yRx)] \rightarrow xRy$ while, assuming R connected, $\sim[yRx] \rightarrow [xRy \vee x = y]$. Upon substitution of our definition of P, Arrow's presentation is undisturbed.) When $xRy \cdot yRz$, we may write '$xRyRz$' or sometimes 'xyz.'

The impossibility theorem, as stated by Arrow and others since, refers to the set T of all transitive, connected, reflexive R on a set S of three or more mutually exclusive 'alternatives,' and to two or more 'voters' or 'individuals' i whose respectively adopted R_i in a given instance are represented by the n-tuple $\rho = (R_{\rho 1}, R_{\rho 2}, \ldots, R_{\rho n})$. The theorem confronts the challenge of finding a function that, conformably to certain conditions, maps the set of all possible ρ into T. We may signify by $\mathbf{R}_\rho$ the image of ρ under a given function into T, whereupon $\mathbf{P}_\rho$ will signify a preference relation included in $\mathbf{R}_\rho$. Among the conditions are 'citizens' sovereignty' and preclusion of a 'dictator.' The political idiom reflects the provenance of the theorem. Condorcet discovered in 1785 from studying electoral processes that when one aggregates transitive relations, one sometimes obtains a relation that fails of transitivity. Arrow envisioned the desired function as a prize of greater theoretical stature, a 'rational' social welfare function for an S of social states. The holy grail of welfare economics, such a function (of which many candidate forms not meeting Arrow's conditions have been proposed) would cut the knot of indecision among infinitely many states, some indistinguishable on efficiency grounds, others approaching distributional equality. We ought to seek, Arrow suggested, some 'composite of the desires of individuals' lest we succumb to a 'Platonic realist' view 'that there exists an objective social good.'[4]

Each ρ (often called a 'profile') provides, as an unordered set would not, a means for allowing a given relation to appear more than once in an argument of a function, for weighting relations defined by the various i, and for defining a notion of dominance. The rationale for referring to 'individuals' who adopt $R_{\rho i}$ is less compelling. Many instances of forming transitive relations are not votes. When alternatives are evaluated according to multiple criteria, there may be formed an $R_{\rho i}$ according to each criterion, but there may be no unique 'author' corresponding to any $R_{\rho i}$. Notwithstanding the heuristic value of anthropomorphism (as in 'a compact set in a metric space may be policed by a finite number of arbitrarily near-sighted policemen'), here it is desirable to lay bare the theorem's generality. As Arrow imagined it, the relevant domain is 'all logically possible' sets of R_i, 'one for each individual.' Without reference to individuals, this domain may be precisely described as the set of all functions $\rho: \mathrm{I} \to \mathrm{T}$ on an index set $\mathrm{I} = \{1, 2, \ldots, n\}$. That is to say that the domain is the n-fold Cartesian product of T.

Arrow's theorem, as stated below in conformity with his axioms I and II, disproves the possibility of the function it describes. With a view to greater generality, to implications for endeavors as diverse as voting and ordering applicants according to desert, we may give the desired function a more general name.

Arrow's Theorem. *Let* O *be the set of binary relations* R *on a set* S *and let* $\kappa = \mathrm{card}\,\mathrm{S}$. *For any* $\mathrm{R} \in \mathrm{O}$, *define* $\mathrm{P} \subset \mathrm{R}$ *as the relation such that* $\mathrm{xPy} \Leftrightarrow [\mathrm{xRy} \cdot \sim(\mathrm{yRx})]$. *For any* $\mathrm{T} \subset \mathrm{O}$, *let* $\rho = (\mathrm{R}_{\rho 1}, \mathrm{R}_{\rho 2}, \ldots, \mathrm{R}_{\rho n})$ *be any element of* T^n, *the Cartesian product* $\prod_{i=1}^{n} \mathrm{T}$ *for the index set* $\mathrm{I} = \{1, 2, \ldots, n\}$. *Define* Φ *as the set of all* ϕ, *each called a* compiler, *in respect of which the following obtain:*

[i] [1] $\phi : \mathrm{T}^n \to \mathrm{B} \subset \mathrm{O}$ *is a function whose value at a given* ρ *is designated* $\mathbf{R}_\rho$, [2] $\mathrm{T} = \{\mathrm{R} \in \mathrm{O} | \mathrm{R}$ *is transitive, connected, and reflexive*$\}$ *and* $\mathrm{B} = \mathrm{T}$, [3] $\kappa \geq 3$ *and* $\mathrm{n} \geq 2$,

[ii] $\forall \rho \in \mathrm{T}^n \cdot \forall \mathrm{x}, \mathrm{y} \in \mathrm{S}([\forall \mathrm{i} \in \mathrm{I}(\mathrm{xP}_{\rho \mathrm{i}}\mathrm{y})] \to \mathrm{x}\mathbf{P}_\rho\mathrm{y})$ [the Pareto unanimous preference condition],

[iii] $\forall\rho, \sigma \in T^n \cdot \forall x, y \in S$, *if* $\forall i \in I(xR_{\rho i}y \Leftrightarrow xR_{\sigma i}y$ *and* $yR_{\rho i}x \Leftrightarrow yR_{\sigma i}x)$, *then* $x\mathbf{R}_{\rho}y \Leftrightarrow x\mathbf{R}_{\sigma}y$ *and* $y\mathbf{R}_{\rho}x \Leftrightarrow y\mathbf{R}_{\sigma}x$ [binary independence of irrelevant alternatives], *and*

[iv] $\sim\exists i \in I|\forall\rho \in T^n \cdot \forall x, y \in S(xP_{\rho i}y \rightarrow x\mathbf{P}_{\rho}y)$ [nondictatorship].

Φ *is empty.*

Since it is crucial for the results below, we recount the proof.[5]

Proof. Given S and I, suppose that Φ is nonempty. Let $\phi \in \Phi$.

[a] We shall say that a subset A of I is rendered *overriding* by ϕ for r over s, signified by $A \rhd \langle r, s\rangle$, if and only if

$$\forall\rho \in T^n([\forall i \in A(rP_{\rho i}s) \cdot \forall i \in I - A(sP_{\rho i}r)] \rightarrow r\mathbf{P}_{\rho}s).$$

Over any given ordered pair, some subset of I is overriding—if no proper subset is overriding, then by the Pareto condition, at least I is. Of I's subsets that ϕ renders overriding for one or another ordered pair, we designate as J the subset of least cardinality (or if several equinumerous sets share that status, we choose any one as J). We may refer to the ordered pair over which J is overriding as $\langle x, y\rangle$. We then select any $j \in J$ and partition J into

$$K = \{j\} \text{ and } L = J - K$$

and let

$$M = I - J.$$

Given that T^n includes every possible n-tuple of T members, we select a $\sigma \in T^n$ such that the following are true in respect of x, y, and some other $z \in S$:

$$xP_{\sigma j}yP_{\sigma j}z,$$

$$\forall l \in L(zP_{\sigma l}xP_{\sigma l}y),$$

and

$$\forall m \in M(yP_{\sigma m}zP_{\sigma m}x).$$

Thus xP_iy for all $i \in J$ while yP_ix for all $i \in M$. Since $J \rhd \langle x, y\rangle$, $x\mathbf{P}_{\sigma}y$.

Consider now the relative positions of y and z under $\mathbf{R}_{\sigma}$. Suppose that $z\mathbf{P}_{\sigma}y$. Then for any $\rho \in T^n$ as to which $yP_{\rho j}z$, $zP_{\rho l}y$, and $yP_{\rho m}z$ for all elements of K, L, and M, respectively—i.e., as to which the relative positions of y and z for all $i \in I$ are the same as under σ—it would follow from premise [iii] that $z\mathbf{P}_{\rho}y(*)$.[6] In such case $L \rhd \langle z, y\rangle$. But $L \rhd \langle z, y\rangle$ cannot obtain because L has one fewer member than J, than which *ex* *hypothesi* there is no smaller overriding set. Hence $\sim[z\mathbf{P}_\sigma y]$. In such case because every $\mathbf{R}_\sigma \in \mathrm{B} = \mathrm{T}$ is connected, $y\mathbf{R}_\sigma z$. From $x\mathbf{P}_\sigma y$ and $y\mathbf{R}_\sigma z$, it follows by transitivity that $x\mathbf{P}_\sigma z$. In such case by [iii] it follows that $x\mathbf{P}_\rho z$ for all $\rho \in \mathrm{T}^n$ as to which $xP_{\rho j}z$, $zP_{\rho l}x$, and $zP_{\rho m}x$ for all elements of K, L, and M, respectively (*). Hence $K \triangleright \langle x, z\rangle$. Since as an overriding set K cannot be smaller than J, it must be that J is a singleton. (Hence L is empty.) Thus we have shown that under ϕ there exists at least one singleton that is overriding for some ordered pair.

[b] Let $B = \{b\} \subset \mathrm{I}$ be such an overriding singleton on some ordered pair $\langle x, y\rangle$. Having defined overriding sets as those whose indexed relations prevail under ϕ on some $\langle r, s\rangle$ against uniform opposition, we now define a subset of the set of overriding sets whose indexed relations prevail regardless of opposition. We shall say that A is *decisive* for r over s, signified by $A \blacktriangleright \langle r, s\rangle$, if and only if

$$\forall\rho \in \mathrm{T}^n([\forall i \in A(rP_{\rho i}s)] \rightarrow r\mathbf{P}_\rho s).$$

We now select a $\sigma \in \mathrm{T}^n$ such that

$$xP_{\sigma b}yP_{\sigma b}z$$

and

$$\forall c \in (\mathrm{I} - B)(yP_{\sigma c}x \cdot yP_{\sigma c}z).$$

Since $B \triangleright \langle x, y\rangle$, $x\mathbf{P}_\sigma y$; by [ii], the Pareto condition, $y\mathbf{P}_\sigma z$; hence by transitivity, $x\mathbf{P}_\sigma z$. By operation of [iii] and the Pareto condition, the foregoing generalizes to $x\mathbf{P}_\rho z$ for all $\rho \in \mathrm{T}^n$ for which there obtains $xP_{\rho b}y$ for b, $yP_{\rho c}x$ for all $c \in (\mathrm{I} - B)$, and $yP_{\rho i}z$ for all $i \in \mathrm{I}$. We observe that $x\mathbf{P}_\rho z$ for all such ρ is thus entailed regardless of the relative positions of x and z under any $R_{\rho c}$ for $c \notin B$. Hence $B \blacktriangleright \langle x, z\rangle$. Thereby has been taken the first step in showing that, as sometimes said, dominance is contagious. If we suppose a τ such that

$$zP_{\tau b}xP_{\tau b}y$$

and

$$\forall c \in (\mathrm{I} - B)(zP_{\tau c}x \text{ and } yP_{\tau c}x),$$

then a repetition of the foregoing form of argument will yield $B \blacktriangleright \langle z, y\rangle$. By repeated interchange of variables and use of the inference $A \blacktriangleright \langle r, s\rangle \rightarrow A \triangleright \langle r, s\rangle$, it may then be deduced that $B \blacktriangleright \langle u, v\rangle$ for every $u, v \in \{x, y, z\}$ when z is any member of S. From this it may be shown by interchange of variables that $B \blacktriangleright \langle u, v\rangle$ for every $u, v \in S$. (The details of this step, omitted here since not germane to the discussion below, address the circumstance that $A \triangleright \langle s, r\rangle$ is not the same as $A \triangleright \langle r, s\rangle$.) Thus when a singleton is overriding for one ordered pair—and by [a] there will always be such a singleton—the singleton is decisive over *all* pairs of S members. Global decisiveness by a singleton constitutes dictatorship. Thus ϕ fails condition [iv]. The supposition that there exists a compiler is false. □

THE ROOTS OF IMPOSSIBILITY

Arrow's theorem engenders surprise because it is not obvious why the seemingly minimal [i]–[iv] should be incompatible. Even as one varies these premises, there swoop in numerous other impossibility results.[7] This leaves if not enhances perplexity about why Arrow's theorem holds. In the following, 'aggregation function' is used to describe any mapping on O^n into O, of which a compiler is a special case. 'Aggregation' is the action of an aggregation function and 'compilation' is the action of a compiler. The value of an aggregation function at an n-tuple of O^n is sometimes called a 'collective relation.'

The effect of premise [iii], entering the proof in the two steps of [a] marked (*), seems to be havoc. Neither Arrow nor Sen state (*) explicitly, and only in proving [b] does either mention the independence of irrelevant alternatives. The case for [iii] is this. At least since Condorcet championed voting by pairwise comparisons, it has seemed reasonable to demand that aggregation should position two alternatives relative to each other correlatively with the way in which the $R_{\rho i}$ position the alternatives with respect to each other. On the other hand, as one criticism would have it, [iii] renders a compiler insensitive to preference intensities. By 'intensities' in this context what are usually implied are real number measurements of utility or of some other purported observable associated with alternatives. Such measurements imply transitivity of indifference,[8] which we shall shortly find reason to doubt, and there is no obvious unit of measure of any such observable common to every $R_{\rho i}$. That aside, it is not [iii] that excludes intensities. They are blocked by [i], which admits into T only information on the relative positions of alternatives.

In another sense 'intensity' may be shorthand for information that [iii] does not allow a compiler fully to recognize. The objection may be posed that any aggregation should assign importance to the fact that whereas x occupies second and y third place in $R_{\rho j}$, x is third and y twentieth in $R_{\rho k}$. Proponents of [iii] deny this. For purposes of compilation, [iii] renders $R_{\rho j}$ and $R_{\rho k}$ indistinguishable as to x and y because it allows only the relative positions of x and y in the aggregated relations to affect their relative positions in $\mathbf{R}_\rho$. It matters not how many z intervene in $R_{\rho j}$ or $R_{\rho k}$ between x and y. This presents the antipode of *ceteris paribus* reasoning: other things need not be equal because other things do not matter. The effect of [iii] is to establish for every $\rho \in \mathrm{T}^n$, as to a given $\langle x, y\rangle$ or $\langle y, x\rangle$, an equivalence class of all σ yielded by any mapping g on $\{\rho\}$ into T^n that is isomorphic on $\langle x, y\rangle$ and $\langle y, x\rangle$ for all i. In (*), the proof exploits such equivalences. To use a chemical analogy, [iii] requires a compiler to burst open the respective $R_{\rho i}$, dissolve all 'bonds' of transitivity among ordered pairs, and rate x and y only by reference to reactions among the liberated $\langle x, y\rangle$'s and $\langle y, x\rangle$'s. The only obvious rating operation as to a given $\langle x, y\rangle$ and $\langle y, x\rangle$ is to count their frequency (perhaps in a weighted fashion) and to install in $\mathbf{R}_\rho$ the more common. An unweighted counting procedure implements *pairwise majority decision* ('PM') in which

$$x\mathbf{R}_\rho y \Leftrightarrow N(xR_{\rho i}y) \geq N(yR_{\rho i}x)$$

where $N(aR_{\rho i}b)$ is the number of i for which $aR_{\rho i}b$. When PM is mentioned hereafter, we may assume that the procedure is executed for all pairs of S elements. (Should two inverse ordered pairs be found in equal numbers, that would compel a nonsymmetric $\mathbf{R}_\rho$ containing both pairs.) However a purported compiler treats pairs, as it reassembles into a new relation ordered pairs that were previously joined by bonds of transitivity, by conforming to [iii] it can yield the handiwork of a virtual voter disdaining transitivity.[9] Part II describes this handiwork.

Recalling that we know every $R_{\rho i}$ to be transitive, we might think of installing, for any pairing that threatens transitivity, some $i \in \mathrm{I}$ as decisive. No pairing would escape the jurisdiction of such transitivity police. This constabulary i would perforce be, or mimic, the dictator whose inevitability we have just deduced (assuming i not powerless under the dictatorship). Premise [ii] may also be expressed as $\cap_{i=1}^{n} P_{\rho i} \subset \mathbf{P}_\rho$. It vindicates unanimity on pairs but does not protect the transitivity in fidelity to which they were formed. Compilation is a casualty of the informational exiguousness imposed by [iii] in the face of the variety of relations presented by [i] when by [ii] unanimity is heeded as to pairs but not as to transitivity whilst overarching counteraction is precluded by [iv].

II

HITCHES, CYCLES, AND HARNESSES

The outcome of aggregation, so Condorcet discovered, may be a *cycle*. As a voting anomaly, a cycle is often mentioned but less often rigorously defined. We first define a *hitch* in respect of any relation R as a subrelation of the form

$$H = \{\langle x, y\rangle, \langle y, z\rangle, \langle z, x\rangle\},$$

also written '$xRyRzRx$,' for distinct x, y, and z. We may then define as a cycle *the union of a hitch and a proper subset of the hitch's inverse.* Hence there occur three forms of three-member cycles:

$$
\begin{aligned}
\mathfrak{C}_{\mathrm{A}}: &\quad H = xPyPzPx,\\
\mathfrak{C}_{\mathrm{Q}}: &\quad H \cup \{\langle x, z\rangle\} = xPyPzIx,\\
\mathfrak{C}_{\mathrm{T}}: &\quad H \cup \{\langle z, y\rangle, \langle x, z\rangle\} = xPyIzIx.
\end{aligned}
$$

The subscript of each $\mathfrak{C}$ corresponds to the name of the weakest condition, as hereafter recounted, that precludes such cycle. Within an antisymmetric relation, when x, y, and z are distinct, $R = P$. In such case $\mathfrak{C}_{\mathrm{A}} = H$ is the lone three-member cycle. The definition of a cycle excludes $H \cup H^{-1} = xIyIzIx$, a relation of total indifference, because when indifference is found in the aggregated relations (as among the alternatives at one level of a tiering), it will not be surprising to find indifference in the collective relation. A cycle is counterintuitive by collision with transitivity as total indifference is not. (Shortly below it is observed that $xIyIzIx$ may be counterintuitive by virtue of transitivity.) In each of the above cycles occur conjunctions of the form $aPb \cdot bRc \cdot cRa$. Transitivity demands that $aPb \cdot bRc \rightarrow aPc$ regardless whether R is P or I. N-member cycles constituted of equivalent strings of P's and I's on larger sets similarly exhibit a failure of transitivity. A failure of transitivity is necessary for a cycle. It suffices for one only if R is connected (since $\sim [aRc] \not\rightarrow cRa$ unless $aRc \vee cRa$). We are generally concerned here with connected $R_{\rho i}$ and $\mathbf{R}_{\rho}$.

We define as a *harness* any relation R on S such that for some hitch H, $H \subset R$ and $H^{-1} \not\subset R$. Typifying the illegitimate output of a purported compiler, a harness R includes one or more cycles not leveled within R by total indifference.

A cycle may be represented by a triangular or circular directed graph whose edges, save for those that are bidirectional, are all oriented clockwise or are all oriented counterclockwise. The directed graph of a harness for $\kappa \geq 4$ resembles that of a non-well-founded (or circular) set. Or again it resembles a system of roads connected by rotaries—evidently the result of attempting an ordering and constructing a snarl. But that is not to imply that in any cycle, R runs viciously round a circle. The directed graph of a cycle need only declare that a relation obtains between the alternatives shown as nodes. Insofar as failure of transitivity is apparent, we cannot assume transitivity anywhere within a cycle. Hence we do not know with respect to $\mathfrak{C}_{\mathrm{Q}}$, for example, that $zIx \cdot xPy \rightarrow zRy$.

CYCLICAL MAJORITIES

The expression 'voting paradox' is a *façon de parler* that describes not a paradox but a striking condition. Let I be partitioned into equinumerous sets *K*, *L*, and *M*. Portraying strict linear orderings, let the following correspond to each member of the respective sets:

$$K : xyz,$$

$$L : yzx,$$

$$M : zxy.$$

By PM, x prevails over y and y prevails over z. By transitivity, this would entail that x prevails over z. But a majority obtains for z over x. The collective relation therefore must be H. For any alternative, there exists another that a majority prefers. To its discoverer, this was nothing if not a contretemps. (Among senses for 'contretemps' in the *Oxford English Dictionary*, 2nd ed., is 'an unexpected hitch.') A sometime votary on voting, Condorcet had insisted before the *Académie Royale des Sciences* on a procedure whereby the candidate elected to membership is one who under majority rule would at ρ defeat every other candidate pairwise. Such a candidate is called a 'Condorcet winner' (or 'CW'). (A CW may also be defined as one who defeats or ties all others, in which case more than one CW may exist when the number of voters is even or if, as Condorcet did not, one allows total tierings among the $R_{\rho i}$. Otherwise a CW is unique.) One may ascertain for any ρ whether a CW exists. Condorcet had now adduced a case in which pairwise comparison is inconclusive. C. L. Dodgson (Lewis Carroll) referred to this predicament as 'cyclical majorities.' In their throes, Dodgson contended, one should declare 'no election' the winner.

Motivated by instances such as elections by plurality voting in which there exists a Condorcet winner who loses, one may define for aggregation functions the following Condorcet winner condition ('CWC'): for every ρ, the CW, if any, tops $\mathbf{R}_\rho$. CWC is satisfied by only one leading voting scheme, sequential pairwise comparison by majority vote. In this procedure one first compares two alternatives, then compares the majority winner to another alternative, then compares the winner thereof to another alternative, and so on until *S* has been canvassed; the CW, if any, will prevail regardless of the sequence of comparison (although below the top of $\mathbf{R}_\rho$, cycles may still occur). Because Condorcet worried about conducting $\binom{\kappa}{2}$ elections—for example, with merely twenty alternatives, there are 190 pairs—he recommended that elections be confined to three candidates per vacancy. The sequential method reduces the number of elections to $\kappa - 1$, but since that may still be inconveniently many, the practical expedient is to elicit an R_i from each i. (This set the stage for Jean Charles de Borda: if eliciting $R_{\rho i}$, why not use more of what they reveal?) But when this sequential scheme encounters any ρ precipitating cyclical majorities, there is no CW. The outcome then mischievously varies with the sequence of comparison. It must also be added that for $n > 2$, if CWC is satisfied, [iii] cannot be.[10]

LATIN SQUARES

Even when $n = 2$, there exists a $\rho \in \mathrm{T}^n$ at which PM yields a cycle.[11] For $n > 2$, given T and any triple Q of alternatives in S, there will always exist a ρ such that I may be partitioned into three sets J_1, J_2, and J_3 of which the following are true: the union of any two J_k includes a majority of i; the $R_{\rho i}$ for the i composing a given J_k coincide on Q in a subrelation U_k; and U_1, U_2, and U_3 form a 3×3 Latin square. A 3×3 Latin square is a matrix in which each element of a triple appears once in each row and once in each column—as in the above example for K, L, and M. At any profile presenting a Latin square, PM produces cyclical majorities and thus a harness.

PM also yields a harness at less monolithic profiles such as may be formed by varying one of the U_k just described to an extent not sufficient to change the relevant pairwise majorities. By inspection, it has been ascertained that the incidence of harnesses increases as κ and n increase. For example, for a domain of n-tuples of strict linear orderings (there being $[(\kappa)!]^n$ such n-tuples), when κ and n are both odd and reach merely 13 and 7, respectively, a harness is yielded by PM at more than half the possible ρ.[12] (If κ and n are even, a separate calculation is needed to account for the effect of ties.) Conversely, for $n > 2$, at any ρ at which PM produces cyclical majorities, there will be found within the R_i of ρ three subrelations whose matrix representation in respect of some triple is a 3×3 Latin square. For given that the $R_{\rho i}$ are all connected, majorities for each of $\langle x, y\rangle$, $\langle y, z\rangle$, and $\langle z, x\rangle$ must be overlapping—each majority comprises more than half the i—and hence each of xyz, yzx, and zxy is found in some $R_{\rho i}$.

THE *EFFET CONDORCET* AND OTHER SYMPTOMS OF CONFLICT

The production of a harness by PM exemplifies a phenomenon pertinent to aggregation functions in general. Borrowing a term sometimes used for the 'voting paradox,' we may call this the *effet Condorcet* and define it as *an instance in which an aggregation function at some* n*-tuple of transitive relations yields a harness.* The *effet Condorcet* saddles us with the handiwork of a virtual voter disdaining transitivity. One convenient way of putting Arrow's theorem is to say that if all other conditions of [i]–[iv] are fulfilled by an aggregation function, transitivity of $\mathbf{R}_\rho$ cannot be fulfilled for all ρ and the *effet Condorcet* is guaranteed. When a purported compiler yields an $\mathbf{R}_\rho$ that fails a codomain membership condition—this betrays an impostor. Why should the impossibility of a compiler be expressed by singling out transitivity of the collective relation? Recognizing a different victim, one could as easily say that when all conditions other than [ii] are fulfilled, [ii] cannot be. It appears that transitivity is the most lamented victim for two reasons. In the first instance, there lurks a possibly artifactual premise [iii] that seems inimical to transitivity of the collective relation in particular. In the second place, PM holds intuitive appeal and happens to fulfill all conditions of [i]–[iv] other than transitivity of the collective relation.[13] Nonetheless there are many interesting tradeoffs among other conditions of [i]–[iv], several of which rise to the surface in the maneuvers that we may now explore.

III

Consider the following dilemma. In an aggregation described by Arrow's theorem, if we insist on transitivity of the collective relation, we must resort to dictatorship or sacrifice some other premise to achieve transitivity. If we do not insist on transitivity of the collective relation, we shall be reduced to cyclic indecision. The first and third maneuvers below seek to grasp this dilemma by its horns. The second defines preemptive subdomains on which the dilemma cannot arise. A fourth strategy identifies patterns in the tallies of $R_{\rho i}$ that allow one to characterize instances in which the dilemma is not posed.

ERSATZ COMPILATION

(A) *Choice.* [a] ARROW'S THEOREM FOR RATIONALIZED CHOICE FUNCTIONS. In decision theory, a *choice function* in respect of S is a function C: $\mathcal{P}S - \emptyset \rightarrow \mathcal{P}S - \emptyset$ such that $C(U) \subset U$. $C(U)$ is called U's 'choice set.' A *rationalization* R of a given C is often said to exist if $C(U) = \{x \in U | xRy$ for all $y \in U\}$. Later we shall find reason to require a slightly different definition of a rationalization, but we may accept this prevalent one for present purposes. A rationalization allows one to say that 'x is R-best in U.' (For a tiering, this will defy the usual understanding that the best element of a set is unique.) By definition, any rationalization generates a choice function. But for some choice functions, no rationalization may be found. An instance is $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x, z\}$, $C(\{y,$

$z\}) = \{y, z\}$, $C(\{x, y, z\}) = \{x, y\}$, the first three equalities of which would establish, by definition of a rationalization, z's entitlement to inclusion in $C(\{x, y, z\})$, which the fourth denies.[14] Let us denote by X the set of choice functions on $\mathcal{P}S - \emptyset$, and by $X_B \subset X$ the set of those for which a rationalization exists in B.

Although some of the premises of Arrow's theorem, [ii] in particular, admit of more than one translation,[15] an alternative version of that theorem, using premises that are translations of [i]–[iv] for choice sets and functions, establishes that there exists no function f_C: $T^n \rightarrow X_B$ for $B = T$.[16] (The version of the theorem proved in part I avoids translational ambiguities while more immediately revealing the theorem's set theoretic ambit.) The impossibility result cannot be avoided by specifying a codomain of choice functions.

[b] A SOCIAL DECISION FUNCTION. But it may be interesting to obtain from aggregation some $\mathbf{R}_\rho$ that generates a choice function C even if neither $\mathbf{R}_\rho$ nor any other relation rationalizing C is transitive. For this purpose, we introduce two conditions weaker than transitivity to which a collective relation might conform. *Quasitransitivity* for nonsymmetric R is the condition $xPy \cdot yPz \rightarrow xPz$. By silence on indifference, quasitransitivity averts the following sorites paradox. When we remove one grain of sand from a heap (a *soros*), a heap remains. Hence the premise seems to follow that if we successively remove one grain at a time, each time the residue will be a heap. But after finitely many removals, there will remain one grain. Either we call that remainder a heap, and thereby contradict common sense, or we do not call it a heap, and thereby contradict the initial premise. Similarly if one stipulates that $xIy \cdot yIz \rightarrow xIz$, suppose that $aIbIcIdIe \ldots mIn$. Failure to discern a preference between consecutive items in this string entails indifference between a and n. But if a and n are quite different, we must choose either to accept what may be an absurd result or to contradict the premise that indifference is transitive. The two paradoxes arise because of the vagueness of 'heap' and 'indifferent.' Poincare was among the first to observe difficulties in transitivity of relations such as indifference.[17] Quasitransitivity, which is to say transitivity of P only, avoids paradox simply by omitting $xIy \cdot yIz \rightarrow xIz$.

The definition of *acyclicity* devolves from that of quasitransitivity by lengthening the string and weakening the conclusion. A given R is acyclic if and only if

$$\forall x_1, x_2, \ldots, x_n \in S([x_1 P x_2, x_2 P x_3, \ldots, x_{n-1} P x_n] \rightarrow x_1 R x_n).$$

'If x_1 is preferred to x_2, x_2 to x_3, and so on until x_n, then acyclicity requires that x_1 be regarded as at least as good as x_n.'[18] Acyclicity may be described as the absence of the antisymmetric cycle $\mathfrak{C}_A$. But acyclicity allows $\mathfrak{C}_Q$ and $\mathfrak{C}_T$, and thus we have acyclic cycles. Since the foregoing definition of acyclicity has become conventional, we may pass by this apparent antinomy. More notable is that quasitransitivity excludes both $\mathfrak{C}_A$ and $\mathfrak{C}_Q$. Quasitransitivity does not exclude $\mathfrak{C}_T$. Only transitivity precludes all cycles, and they of any length.

Let us suppose in respect of $U = \{x, y, z\}$ three choice functions C_A, C_Q, and C_T with rationalizations $\mathfrak{C}_A$, $\mathfrak{C}_Q$, and $\mathfrak{C}_T$, respectively. On inspection of

$$\mathfrak{C}_A = \{\langle x, y\rangle, \langle y, z\rangle, \langle z, x\rangle\},$$

we find acyclicity lacking and $C_A(U)$ empty. For acyclic

$$\mathfrak{C}_Q = \{\langle x, y\rangle, \langle y, z\rangle, \langle x, z\rangle, \langle z, x\rangle\},$$

$C_Q(U) = \{x\}$, and for acyclic

$$\mathfrak{C}_T = \{\langle x, y\rangle, \langle y, z\rangle, \langle z, y\rangle, \langle x, z\rangle, \langle z, x\rangle\},$$

$C_T(U) = \{x, z\}$. This suggests a gain from enlarging an aggregation function's codomain. Let

$$\mathrm{O}_A = \{R \in \mathrm{O} | R \text{ is reflexive, connected, and acyclic}\}.$$

Sen showed that

[Sen I] For finite S, any $R \in \mathrm{O}_A$ generates a choice function.

[Sen II] For finite S, there exists, conforming to [i]–[iv] except that $\mathrm{B} = \mathrm{O}_A$, a *social decision function* ('SDF') $f_D\colon \mathrm{T}^n \to \mathrm{O}_A$.[19]

This suggests the strategy of finding what one considers a suitable SDF. The acyclic relation yielded thereby will furnish a collective choice function.

It might be objected that it is no feat to assure the existence of a choice function insofar as the axiom of choice asserts the existence of a choice function $F\colon \mathcal{P}S - \emptyset \to S$, $F(U) \in U$, even for infinite S. The social choice theorist might reply that X is defined to contain functions $C\colon \mathcal{P}S - \emptyset \to \mathcal{P}S - \emptyset$, $C(U) \subset U$, i.e., functions that select not singletons but subsets. To this it may be rejoined that the axiom of choice asserts the existence of such a C for any $\mathcal{P}S - \emptyset$ merely by setting $C(U) = \{F(U)\}$. What makes the coupling of Sen I and II significant is not the existence of some choice function, but the existence of one furnished by a relation derived from an $R_{\rho i}$ conformably to all but one condition of Arrow's theorem.

We may compare possible mappings and the aftermoves:

$$\phi : \mathrm{T}^n \to \mathrm{B}.$$

$$f_{\mathrm{C}} : \mathrm{T}^n \to \mathrm{X}_{\mathrm{B}} : \mathcal{P}S - \emptyset \to \mathcal{P}S - \emptyset.$$

$$f_{\mathrm{D}} : \mathrm{T}^n \to \mathrm{O}_{\mathrm{A}} \to \mathrm{X} : \mathcal{P}S - \emptyset \to \mathcal{P}S - \emptyset.$$

It is possible to attempt a compilation, discover an *effet Condorcet*, ascertain that the collective relation happens to be acyclic, and then generate a choice function from that collective relation. The problem is that we may not find a choice function that seems helpful.

The yield from employing an SDF falls short of an ordering. Consider the example of an SDF by which Sen proved Sen II. First let $\mathbf{D}_\rho$ be the relation of Pareto superiority defined as follows:

$$x\mathbf{D}_\rho y \Leftrightarrow \forall i(xR_{\rho i}y) \cdot \exists i(xP_{\rho i}y).$$

(By comparison to this condition, [ii] is sometimes called the 'weak Pareto condition.') $\mathbf{D}_\rho$ is transitive. (If $x\mathbf{D}_\rho y$ and $y\mathbf{D}_\rho z$, then $\forall i[xR_{\rho i}y] \cdot \exists i[xP_{\rho i}y]$ and $\forall i[yR_{\rho i}z] \cdot \exists i[yP_{\rho i}z]$. Since every $R_{\rho i}$ is transitive, it follows that $\forall i[xR_{\rho i}z]$. From $\exists i[xP_{\rho i}y]$ and $\forall i[yR_{\rho i}z]$ it also follows that $\exists i[xP_{\rho i}z]$. Thus $\forall i[xR_{\rho i}z] \cdot \exists i[xP_{\rho i}z]$, or $x\mathbf{D}_\rho z$.) $\mathbf{D}_\rho$ is obviously also antisymmetric. But on a given set, $\mathbf{D}_\rho$ may not be connected. Hence on T^n, any $f(\rho) = \mathbf{D}_\rho$ fails to qualify as an SDF. Sen's exhibited SDF instead is a function g that yields the collective relation of Pareto noninferiority:

$$g : \mathrm{T}^n \to \mathrm{O}_{\mathrm{A}},\ g(\rho) = \mathbf{R}_\rho \text{ where } x\mathbf{R}_\rho y \Leftrightarrow \sim[y\mathbf{D}_\rho x].$$

Here $x\mathbf{R}_\rho y$ if and only if y is not Pareto superior to x. It is apparent that any such $\mathbf{R}_\rho$ is connected on any set, quasitransitive and hence acyclic, reflexive, and hence contained in O_{A}.

Suppose the conjunction $xR_{\rho i}y \cdot yP_{\rho j}x$ for merely one i and one j. Neither x nor y is Pareto superior to the other, and g yields $x\mathbf{R}_\rho y$ and $y\mathbf{R}_\rho x$, i.e., $x\mathbf{I}_\rho y$. This reveals two problems. First, g will yield $x\mathbf{I}y\mathbf{I}z$ at any ρ presenting a 3×3 Latin square. At such ρ, under $C_{g(\rho)}$ derived from $g(\rho)$, for some $\{x, y, z\}$ we shall have $C_{g(\rho)}(\{x, y, z\}) = \{x, y, z\}$. In welfare economics occur infinitely many alternatives lacking any Pareto superior. These maximal elements of S with respect to the partial ordering $\mathbf{D}_\rho$ are said to be 'Pareto optimal' or 'efficient,' and $g(\rho)$ is indifferent among them. The multiplicity of optima helped fuel the quest for a social welfare function—a means of selecting, by reference to individual preferences, an *optimum optimorum* (or perhaps a nonoptimum instead). All it takes for *every* alternative to lack a Pareto superior, for all alternatives to be viewed indifferently under $g(\rho)$, is sufficient variation in the $R_{\rho i}$ such that $xR_{\rho i}y \cdot yP_{\rho j}x$ in the case of each $\{x, y\}$ for some i and j. $C_{g(\rho)}$ is notably undiscriminating. Worse, 'it is not at all clear,' as Sen writes, 'that other examples will be more appealing.'[20] A linear ordering guarantees a choice function yielding singletons, but any acyclic or quasitransitive collective relation that fails to be transitive may generate only a weakly selective choice function. Often $C(U)$ will be some large subset of U or simply U. The explanation for this weak selectivity lies in the inability of an SDF, even if the $R_{\rho i}$ are antisymmetric, to guarantee more than a nonsymmetric collective relation. Tierings may be suffused with ties. In respect of any hope that the collective relation be antisymmetric, we observe that given antisymmetry, the distinctions among acyclicity, quasitransitivity, and transitivity collapse. Thus to demand that the output of an SDF be antisymmetric would be to require the transitivity that Arrow's theorem withholds.

Second, in g we observe veto power of each i. Regardless of majorities, if $xP_{\rho i}y$ for even one i, then $x\mathbf{R}_{\rho}y$ is assured and $y\mathbf{P}_{\rho}x$ precluded. In general, so it has been shown, any SDF yielding quasitransitive but not transitive relations makes some subset of I an oligarchy. An oligarchy is a set each member of which holds a veto and whose members together are decisive.[21] A small oligarchy has been said to 'concentrate the power of decisiveness between all pairs of alternatives in a small group'; under a large oligarchy, 'a multiplicity of weak pairwise dictators will make the choice functions nonselective.'[22] Should oligarchs not be unanimous, then by virtue of their inherent veto power, ties may abound. When $\kappa \geq 4$, $\kappa > n$, and the collective relation is merely acyclic, at least one i holds veto power over some significant subset of pairwise comparisons.[23]

Thus the embrace of a choice function challenges the second horn of the dilemma only by dulling it. An SDF trades cyclic indecision for weak selectivity, dictatorship for oligarchy. When a protégé asks a mentor to rank the ingredients for professional success, to be informed of the 'equally best' may be unsatisfying. *Summa sedes non capit duos* ('the highest seat does not hold two'). Even when a relatively large choice set is the goal, as when a college admits thousands of applicants, there may be such curiosity whether the choice function is consistent—in the operation of a choice function, 'consistency' is the equivalent of a relation's transitivity—as to importune a function that is significantly selective. Should each $R_{\rho i}$ be the result of applying some single criterion and should some SDF install an oligarchy, it may be difficult to explain that some criteria effectively do not matter.

(B) PM *as a Subcompiler on a Restricted Domain*. A typical aggregation like an election presents merely one profile. That is not to imply that a domain smaller than T^n may suffice for a compiler, for one never knows what profile will appear. The observation does motivate an effort to characterize profiles at which a given method of aggregation does not yield harnesses. It is a contingent possibility that the tally under PM of various R_i presented in a given ρ will yield a transitive collective relation, and in (D) we shall see a description of that contingency. But first there is a notable preemptive move. This is to identify a subset of T such that no possible n-tuple of its elements could produce a Latin square. (The possible n-tuples form a larger set than the n-membered permutations of T since a given R may appear more than once in an n-tuple.) Contrasting with enlargement of the codomain in (A), this categorical move would describe, and suggest the possibility of contriving or fortuitously encountering, a restricted domain.

Let us denote by Γ_X any subset of T that satisfies condition X. If for some Γ_X and for some $D \subseteq \Gamma_X{}^n \subset T^n$, there exists an $f\colon D \rightarrow B$ satisfying all conditions of [i]–[iv] other than $D = T^n$, let f be known as a *subcompiler*. It may be shown that PM is a subcompiler under the following two conditions.[24] The first defines Γ_{VR} as the set of those $R \in T$ such that the following condition, *value restriction* ('VR'), obtains: for every $Q = \{x_k\} \subset S$, $k = 1, 2, 3$, either there exists some x_k that no $R \in \Gamma_{VR}$ places first in Q, or there exists some x_k that no $R \in \Gamma_{VR}$ places between the other two x_k in Q, or there exists some x_k that no $R \in \Gamma_{VR}$ places last in Q. (Γ_{VR} may exhibit one of these position-avoidance characteristics for Q_i, another for Q_j.) The second condition, *odd parity* ('OP'), specifies a domain $D \subseteq \Gamma_{VR}{}^n$ such that for every $\rho \in D$ and every Q, the number of $R_{\rho i}$ concerned about Q is odd. $R_{\rho i}$ is said to be concerned about Q if $R_{\rho i}$ does not evince total indifference as to Q (i.e., if $x_l P_{\rho i} x_m$ for at least one $\{x_l, x_m\} \subset Q$). OP requires attention to the R_i presented in a particular $R_{\rho i}$. VR is satisfied only if the stated position-avoidance condition holds regardless which and how often any $R \in \Gamma_{VR}$ appear in any $R_{\rho i} \in \Gamma_{VR}{}^n$.

As earlier noted, PM satisfies all relevant conditions of [i]–[iv] save for transitivity of the collective relation. The proof that PM under VR-OP is a subcompiler therefore need only show that under VR-OP the elements of PM's range are transitive. Suppose that VR-OP obtains and that $Q = \{x_1, x_2, x_3\}$ is an arbitrary triple from S. If x_1 is not best in Q under any $R \in \Gamma_{VR}$, then any instance of either $x_1 R_{\rho i} x_2 R_{\rho i} x_3$ or $x_1 R_{\rho i} x_3 R_{\rho i} x_2$ must be an instance of $x_1 I_{\rho i} x_2 I_{\rho i} x_3$. Sen showed that if $x_1 R_{\rho i} x_2 R_{\rho i} x_3$ and $x_1 R_{\rho i} x_3 R_{\rho i} x_2$ are $x_1 I_{\rho i} x_2 L_{\rho i} x_3$ for all $R_{\rho i}$ on Q, then a hitch may occur only if the number of $R_{\rho i}$ concerned about Q is even.[25] (Sen also showed that for PM and some other aggregation functions, VR suffices to assure a quasitransitive collective relation, and hence a choice function.[26] Thus we might say that odd parity is needed to avert $\mathfrak{C}_T$.) Similarly if for all $R_{\rho i}$ on Q, x_2 or x_3 is not best, or if any of the three is not worst or not between the other two, then in each case the two subrelations that could contradict the respective condition must each be instances of $x_1 I_{\rho i} x_2 I_{\rho i} x_3$, and under that circumstance, a hitch cannot occur unless the number of $R_{\rho i}$ concerned about Q is even. Since under OP the number of concerned $R_{\rho i}$ is odd, a hitch is impossible. The connectivity of the respective $R_{\rho i}$ insures that the collective relation generated by PM will be connected. As earlier noted, failure of transitivity in a connected relation entails a cycle. Hence via the contrapositive of that entailment, the absence of a hitch entails transitivity of the collective relation.

For finite S, PM under VR-OP assures a CW. When the collective relation produced by PM is transitive and connected, as VR-OP assures, there cannot fail to be a CW (though there may be more than one). But to exhibit VR-OP, a domain may include only n-tuples satisfying OP and consisting only of R whose every assortment as an n-tuple satisfies the position avoidance condition of VR. For that reason the fortuity of VR-OP will be unusual. The probability of odd parity for merely one triple may be no better than 0.5, to say nothing of all triples. Were one attempting to contrive VR-OP, one could assure OP by (a) confining D to antisymmetric relations and thereby precluding indifference, or allowing no more than two alternatives at any level of a tiering (but if two may tie, why not three?), thereby in either case making all $R_{\rho i}$ concerned, and (b) arranging for the number of i to be odd. To achieve (b) in an electoral setting, one might empower someone to vote only if the total number of votes would otherwise be even. Since that would confer decisiveness on such designee, instead one might discard one randomly selected ballot. But VR will remain elusive. In some contexts that may be a mercy. VR is exemplified by an election in which candidates array indisputedly from left to right along a one-dimensional scale and all voters unfailingly vote according to candidate scale position. In such case no voter rates the centrist candidates worst.

There is a condition under which PM will constitute a subcompiler without OP, but that condition is seemingly more unusual than VR. *Extremal restriction* ('ER') is the condition that if for any $i \in \mathrm{I}$, $x_1 R_{\rho i} x_2 R_{\rho i} x_3$, then for any other $j \in \mathrm{I}$ such that $x_3 R_{\rho j} x_1$, it must be that $x_3 R_{\rho j} x_2 R_{\rho j} x_1$. On any $D' = \Gamma_{ER}{}^n$ where Γ_{ER} satisfies ER on every triple $Q \subset S$, PM is a subcompiler. Sen has proved this without reference to a Cartesian product $\Gamma_{ER}{}^n$ but equivalently by reference to a set of relations and their 'assignment . . . over some number of individuals.'[27]

(C) *The Borda Count*. Considering the earlier observation that [iii] effectively installs virtual voters who flout transitivity, and given that dominance could not reach its contagious dictatorial extreme but for (*)'s generalizing effect in [b] of the proof, one might be inclined to reject [iii]. Thereby one would repudiate the first horn of the dilemma that we have been considering. Arrow's theorem denies the conjunction of transitivity and nondictatorship only if the other premises obtain. Waiving [iii], aggregation could be accomplished conformably to [i], [ii], and [iv] by the Borda count. Advocated in 1770 in an address to the *Académie* by Borda (though first formulated in 1434 by Nicolaus Cusanus as a scheme for electing the emperor of the Holy Roman Empire), the Borda count is now used in society elections (as by the American Mathematical Society) and polls (as in college football).

Seizing on the failure of plurality voting to satisfy the CWC, Borda insisted that account be taken of voters' $R_{\rho i}$, not merely of the one candidate that they respectively consider best. The Borda count assigns $\kappa - j$ points to the jth-listed alternative in $R_{\rho i}$. (If $R_{\rho i}$ places m alternatives in one position, then of the total points associated with that position and the $m-1$ positions immediately below it, each tied alternative receives a $1/m$ share.) For each alternative, the points thus awarded for all $i \in \mathrm{I}$ are summed. (The total happens to equal the total points that the alternative would receive in pairwise comparison with each other alternative were the preferred to receive one point and the other none.) The totals for alternatives are compared, and an ordering thus follows on S by the natural ordering on ω. No harness can occur. Nor can a dictatorship. Apart from the fact that it too fails the CWC while violating [iii],[28] the Borda count's famous defect is manipulability by untruthful voting. All nondictatorial voting schemes are manipulable,[29] but the Borda count remains singular for the transparency by which the method of its successful manipulation is revealed. In a football poll, should coach Grid esteem team x first and y second, he may so record them, but if instead Grid lists x first and places y far below second, Grid will improve the chances for, and under favorable circumstances may assure, x's ascendancy. By contrast, absent clairvoyance it may not even be computationally feasible to manipulate the single transferable vote.[30] The Borda count may also be manipulated by specifying some proper subset of S as the set of candidates, or by voter truncation of preference lists. Dropping from candidacy even an alternative known to finish well below first may sometimes alter who does finish first.

(D) *Specifying Cyclic Contingencies under* PM. Instead of categorically envisioning as in (B) a sanitized Γ_X generating a domain so small that there is no peradventure of a Latin square, it is possible to characterize the conditions under which a harness will occur when PM operates on all of T^n. Clearly if the same $R_{\rho i}$ is declared by more than half the i, $\mathbf{R}_\rho$ will be that $R_{\rho i}$, and no harness occurs. But except when κ is small, PM ushers in cycles whenever the $R_{\rho i}$ vary even slightly.[31] Profiles may be elucidated by vectorial or graphical representation. Consider first a subrelation of a single $R_{\rho i}$ that positions the elements of a triple $Q = \{x_1, x_2, x_3\}$. One may represent the subrelation as a three-dimensional vector $\mathbf{r}_{\rho i Q}$ whose components are the respective tallies of x_1 over x_2, x_2 over x_3, and x_3 over x_1.[32] Here the 'tally of x_i over x_j' is stipulated as 1 when $\langle x_i, x_j\rangle \in R_{\rho i}$, as -1 when $\langle x_j, x_i\rangle \in R_{\rho i}$, and as 0 when both such ordered pairs belong to $R_{\rho i}$. The x_k may be interchanged and the orientation reversed so long as a consistent scheme is followed. (Alternatively one may construct a directed graph with alternatives as nodes and tallies as edge values.) It may be observed that $R_{\rho i}$ will generate a harness on Q if and only if $\mathbf{r}_{\rho i Q}$'s components are all nonnegative or all nonpositive, a condition that we may call the 'cycle condition.' One may readily verify that when there occur cycles of the forms $\mathfrak{C}_A$, $\mathfrak{C}_Q$, and $\mathfrak{C}_T$, the cycle condition is met. For the transitive $R_{\rho i}$ of T, the cycle condition will fail. For example, when $x_1 P_{\rho i} x_2 P_{\rho i} x_3$ and $x_1 P_{\rho i} x_3$, $\mathbf{r}_{\rho i Q} = (1, 1, -1)$.

The key point is that every $\mathbf{r}_{\rho i Q}$ may be uniquely represented as the sum of a 'cyclic' vector and an orthogonal 'cocyclic' vector, the former always nonzero. As the curl of a vector field is the local tendency to produce a whirlpool, the cyclic vector of a relation is the local tendency to produce a cycle. Denoting the sum of a given $\mathbf{r}_{\rho i Q}$'s tally components as 'spin,' the three components of the cyclic vector are each one-third the spin. The cocyclic vector's components are respectively two-thirds the point spreads for x_1 over x_2, x_2 over x_3, and x_3 over x_1, the points being those won in a Borda count according to $R_{\rho i}$. Thus this scheme only avails for $R_{\rho i}$ that establish a positioning (as defined below) of the elements of each Q, for which reason the Borda count is aptly called a 'positional' voting scheme. For a given $R_{\rho i}$, the points will be 1 for first, 0 for second, and -1 for third place (subject to the provision for ties earlier noted), and thus the point spreads will range from -2 to 2. When many $R_{\rho i}$ are aggregated, the sum of the cocyclic components may be any integer.

In a transitive relation such as $R_{\rho i}$, the cocyclic vector 'masks' the cyclic vector. The mischief of the *effet Condorcet* is always latent. When many $R_{\rho i}$ are aggregated, their corresponding $\mathbf{r}_{\rho i Q}$ are added. It is thereupon contingent whether the respective cyclic and cocyclic vectors underlying the $\mathbf{r}_{\rho i Q}$ will reinforce and cancel in such a way that the resultant $\mathbf{r}_{\rho Q}$ will fulfill the cycle condition. In $\mathbf{r}_{\rho Q}$, which corresponds with the collective relation, the component tally of x_i over x_j will be the number of $\langle x_i, x_j\rangle$

occurring among the $R_{\rho i}$ less the number of $\langle x_j, x_i \rangle$. The cyclic contingency may be precisely specified. First, for each possible subrelation on Q (of which, allowing for ties, there are 13), one may identify the cyclic and cocyclic vectors. Second, given any ρ and Q, one knows which of the subrelation types on Q appear in the $R_{\rho i}$ and in what numbers. One thereby knows $\mathbf{r}_{\rho Q}$. Should $\mathbf{r}_{\rho Q}$ be found to fail the cycle condition for all possible Q, the collective relation will contain no three-member cycle. One may therefore identify conditions on the proportionate representation of discrete subrelation types within the $R_{\rho i}$ such that for all Q, $\mathbf{r}_Q$ fails the cycle condition.

While this explanation of cyclic contingencies allows predictions, it shares with VR-OP the limitation that there appears no obvious way to exploit it except by domain restrictions vulnerable to characterization as *ad hoc* contrivances to avoid the cycle condition.

IV

ROOM FOR INDIFFERENCE

Many relations of striking propinquity to those governed by Arrow's theorem avoid the Scylla of dictatorship and the Charybdis of cyclic indecision by slipping through a curious gap in the theorem. According to premise [i][2], T consists of all and only weak linear orderings and weak total tierings. Other linear orderings and total tierings, which is to say those that are irreflexive or nonreflexive, are absent from T and left unfettered by the theorem. What explains this?

Arrow's nomenclature obtrudes here. Arrow introduced an element of T as a 'weak ordering,' a term he soon shortened to 'ordering.' For Arrow and economists since, an ordering may be either nonsymmetric or antisymmetric.[33] As $\leq$ is to $<$ on $\mathbb{R}$, said Arrow, so is a nonsymmetric ordering to an antisymmetric one. For a set theorist, this assimilation is confounding: $\leq$ and $<$, antisymmetric both, differ in a contingent feature of orderings ($\leq$ is reflexive, $<$ irreflexive), while 'nonsymmetric ordering' is self-contradictory inasmuch as an ordering must be antisymmetric. (By Russell's definition, an ordering must even be asymmetric.[34]) In the set theorist's argot, T comprises connected preorderings.[35] Nevertheless subsequent presenters of the theorem have followed Arrow, asserting the theorem for T and describing it as the set of orderings.[36] Upon encountering in prose a theorem said to govern all orderings, one might assume, though as it happens incorrectly, that the theorem governs all linear orderings in the set theoretic sense. After all, the subrelations forming Latin squares, so frequently exhibited for the *effet Condorcet*, are antisymmetric.

Motivation for including weak total tierings in T is at least clear. Indifference has long been regarded by economists as the sensible description of a consumer who esteems two or more distinct relata equally. Indifference curves, level curves of utility functions or, more fundamentally, depictions of equivalence classes under the relation I, are used to describe behavior, equilibrium, and conditions of Pareto optimality. For Arrow it seemed evident that his binary relations 'can and must' reflect indifference.[37] This motivated predicating a theorem of R_i of which an economist may say, '$x R_i y$ just in case x is preferred *or indifferent* to y.'

Seeking to understand the exclusion of irreflexive and nonreflexive relations from T, we might conjecture that the sharp discriminations of strict linear orderings enable a compiler. But this would not seem to mesh with a phenomenon noted shortly ago, the advent of cycles when the $R_{\rho i}$ vary even slightly, nor with a further result that the more prevalent is indifference in the $R_{\rho i}$, the less likely are cycles in the collective relation.[38] Were the instant conjecture true, Arrow's theorem could be circumvented by any weak ordering purged of pairs $\langle x, x\rangle$. (To any weak ordering corresponds a unique strict ordering.) We refute the conjecture by adverting to the ubiquity, among n-tuples of strict linear orderings just as in T^n, of the σ's and τ's adduced in the proof of Arrow's theorem. These scuttle the possibility of a compiler.

Calling a transitive and connected relation a *positioning*, we arrive at the following.

A GENERALIZATION

Impossibility Theorem for Positionings. *Let* $\Psi = \{\mathrm{R} \in \mathrm{O} | \mathrm{R}$ *is transitive and connected*$\}$. *Let terms be defined as in Arrow's theorem except that in* [i] *of the definition of* Φ, [2] *is replaced by any one of the following:*

[2a] $\mathrm{T} = \{\mathrm{R} \in \Psi | \mathrm{R}$ *is antisymmetric and reflexive*$\}$, $\mathrm{B} = \Psi$,
[2b] $\mathrm{T} = \{\mathrm{R} \in \Psi | \mathrm{R}$ *is antisymmetric and nonreflexive*$\}$, $\mathrm{B} = \Psi$,
[2c] $\mathrm{T} = \{\mathrm{R} \in \Psi | \mathrm{R}$ *is nonsymmetric and reflexive*$\}$, $\mathrm{B} = \Psi$,
[2d] $\mathrm{T} = \{\mathrm{R} \in \Psi | \mathrm{R}$ *is nonsymmetric and nonreflexive*$\}$, $\mathrm{B} = \Psi$, and
[2e] $\mathrm{T} = \{\mathrm{R} \in \Psi | \mathrm{R}$ *is asymmetric*$\}$, $\mathrm{B} = \Psi$.

Φ *is empty.*

Proof. For each alternate version of premise [i][2], the proof of the result is identical to the earlier proof of Arrow's theorem. □

To see why the earlier proof works here, we first observe that the proof nowhere depends on reflexivity of any R in T or B. The proof employs, so far as T and B are concerned, only (a) various $P \subset R \in$ O, which P happen to be asymmetric and irreflexive regardless what R may be, and (b) $R \neq P$ in two inferences (viz., $\sim [z\mathbf{P}_\sigma y] \rightarrow y\mathbf{R}_\sigma z$ and $[x\mathbf{P}_\sigma y \cdot y\mathbf{R}_\sigma z] \rightarrow x\mathbf{P}_\sigma z$) from the definitions of P, connectivity, and transitivity alone. Hence upon revising [i][2] by enlarging the codomain to Ψ, we obtain a stronger result sustained by the same proof. It is stronger not by virtue of any notion that reflexivity is a demanding condition, but by blocking a seeming route of escape. Arrow's proof of his theorem for $n = 2$ refers to indifference,[39] but the proof given earlier for $n \geq 2$ is indifferent to indifference. This proof nowhere requires for any R even the possibility that $xRy \cdot yRx$ for $x \neq y$. The definition of each T of [2a]–[2e] contains, in place of reflexivity in the definition of elements of the original T, a pair of relational attributes drawn respectively from the sets {reflexive, nonreflexive, irreflexive} and {antisymmetric, nonsymmetric, asymmetric}. (There are five feasible pairs; four other pairs are infeasible because asymmetry entails irreflexivity and transitive irreflexivity entails asymmetry.) For each of these T, T^n contains n-tuples of the same form as the σ's and τ of the proof.

It was early observed 'how *economic* Arrow's impossibility theorem is. Relax any of his restrictions and the result collapses.'[40] But we have found that reflexivity of $\phi(\rho)$ is an unnecessary premise. As a membership condition of T, reflexivity is unnecessarily restrictive insofar as the generalization reveals that partitioning Ψ according to any of the attributes of reflexivity, nonreflexivity, irreflexivity, antisymmetry, and nonsymmetry carves out no domain on which a compiler exists. Apart from the partition of Ψ formed by the T's of [2a]–[2e], one may identify many other Ψ subsets that admit of no compiler. Any T big enough such that T^n includes the σ's and τ of the proof will do.[41]

Why should Arrow and followers have confined their gaze to reflexive relations? Indifference requires for distinct x and y only nonsymmetry, and nonsymmetry is compatible with nonreflexivity and reflexivity. (xRx is required, by transitivity, only in cases of $xRy \cdot yRx$.) The answer, according to Arrow, is that it is necessary to capture indifference over identicals: 'we ordinarily say that x is indifferent to itself for any x.'[42] 'This requirement,' adds Sen, 'is so mild that it is best looked at as a condition, I imagine, of sanity rather than of rationality.' Why should sanity compel reflexivity? We do not count inability to say '$5 < 5$' as a hiatus in rational thinking. Sen answers that without reflexivity, a choice function is impossible.[43] Sen adverts in this claim to the circumstance that he and Arrow have already defined the choice set of U solely in terms of a rationalization R according to the following expression:

$$C(U, R) = \{x \in U | \forall y \in U(xRy)\}.$$

According to this, the choice set of every U (including even $\{x\}$) will indeed be empty unless xRx.

But, *pace* Arrow and Sen, we need not be painted into this corner, into what Berkeley called 'one of those difficulties which have hitherto amus'd philosophers ... entirely owing to our selves—that we have first raised a dust and then complain we cannot see.'[44] Instead we may interpose a simpler definition offered in part III, namely, that a choice set is merely any set yielded by a choice function, which in turn we define as any function $C: \mathcal{P}S - \emptyset \rightarrow \mathcal{P}S - \emptyset$ such that the image of each argument is a subset of the argument (i.e., such that $C(U) \subset U$). The existence of a function $C: \mathcal{P}S - \emptyset \rightarrow \mathcal{P}S - \emptyset$ does not imply or presuppose any other relation. Moreover, for all purposes for which a rationalization remains of interest—such as the choice version of Arrow's theorem—a rationalization of C may now be defined as any R such that

$$C(U) = \{x \in U | \forall y \in U(xRy \vee [x = y])\}.$$

(An alternative condition is $C(U) = \{x \in U | \forall y \in U - \{x\}(xRy)\}$ provided that $C(\{x\}) = \{x\}$.) This definition does not require (though it permits) R to be reflexive.[45] The only prerequisite of a rationalization is membership in O. Thus even for the narrow purpose of modeling consumer behavior, no compelling reason arises to stipulate reflexivity.

We also discover that by dropping reflexivity from the definition of O_A, we obtain a stronger version of Sen I. When a choice function is now understood as any $C: \mathcal{P}S - \emptyset \rightarrow \mathcal{P}S - \emptyset$, $C(U) \subset U$, the stronger version follows from arguments in the proof of the original Sen I based on connectivity and acyclicity alone. We may (but need not) similarly redefine O_A for purposes of Sen II, thereby enlarging f_D's codomain. More importantly, we may iterate Sen II as to an f_D for each T in [2a]–[2e]. This follows because the example proving Sen II requires nothing of the $R_{\rho i}$ except transitivity. The generalized Sen I and II thus follow as modified under their original proofs unaltered. All other results of part III obtain for the T and B of [2a]–[2e] since we made no use in part III of any relational attributes other than transitivity and connectivity. Similar generalizability obtains for Debreu's theorems. Building upon set theoretic results of Cantor and Birkhoff but introducing topological concepts, Debreu showed, in a theorem applicable to any separable connected topological space S, and in a second theorem applicable to any subset S of a separable metric space, that for any 'continuous' connected preordering R on such an S, there exists (uniquely up to an increasing transformation) a continuous order homomorphism on S under R into $\mathbb{R}$ under $\geq$—that is, a utility function.[46] Debreu's proofs do not require reflexivity as a premise. Hence in restating his theorems, we may replace 'connected preordering' with 'positioning.' Still another theorem of Sen's reprises Arrow's impossibility result, this time for any mapping into Arrow's original T from the n-fold Cartesian product of $\{u_i | u_i$ is a utility function representing an $R_i \in \mathrm{T}\}$.[47] Reflexivity may without cost be dropped in respect of both the codomain and the u_i.

Connected preorderings may not be the only positionings adopted by *homo oeconomicus*, the agent whose conative features it has been a principal goal of many theories to model. We encounter ballots and any number of other lists conveying irreflexive and nonreflexive positionings—as in surveys of preferences among goods, nonmeasure rankings of applicants for admission to college or for employment, or other nonmeasure positionings of alternatives by multiple evaluators or criteria of value or desert. The impossibility theorem for positionings establishes for any set of at least three members the impossibility of assuring as little as a total tiering when aggregating as much as strict linear orderings. For each set of positionings specified in the theorem, there exists no mapping that by nondictatorial pairwise comparisons respecting unanimity compiles multiple positionings into one.

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NOTES

1 Prefatory note to chapter 1, Arrow 1983, pp. 1–2.

2 As may be found, e.g., in Royden 1988, pp. 24–25.

3 Citing Tarski, he acknowledged that 'strictly speaking, a relation is said to be connected if Axiom I holds for $x \neq y$' (Arrow 1963, p. 13, n. 6).

4 This dichotomy, at Arrow 1963, pp. 22–23, has subsequently been exploded by the articulation of non-Platonic preference-independent accounts of individual and social good.

5 This is an adaptation for the above formulation of the theorem of the proof in Sen 1970, pp. 28, 41–46, which is based upon Arrow's second of two proofs (the second correcting an error in the first) in Arrow 1963, pp. 96–100.

6 Parenthetical asterisks mark steps for future reference.

7 Many such theorems are presented in Kelly 1978.

[8] Research motivated by this realization is described in Roberts 1989.

[9] Saari 1998 describes the process as tantamount to excluding everyone except 'primitive' voters.

[10] The proof is recounted in Fishburn 1972, p. 151, and Taylor 1995, pp. 114, 120, 124–126, the latter also showing that the sequential method fails [ii].

[11] When $xP_{\rho 1}yP_{\rho 1}z$ and $zP_{\rho 2}xP_{\rho 2}y$, PM yields $x\mathbf{P}_{\rho}y$ and, because $aP_{\rho 1}b \cdot bP_{\rho 2}a \rightarrow a\mathbf{I}b$ (as proved in Arrow 1963, pp. 50–51), $y\mathbf{I}_{\rho}z$ and $z\mathbf{I}_{\rho}x$, i.e., $\mathfrak{C}_{\mathrm{T}}$.

[12] Gehrlein 1983 discussed in Kelly 1988, pp. 20–22.

[13] This is shown in Arrow 1963, pp. 46–48.

[14] This example is owed to Kelly 1978, p. 20.

[15] The possibilities are reviewed in Sen 1997, pp. 16–17.

[16] For the theorem and proof, see Fishburn 1972, p. 42, and Kelly 1978, p. 39.

[17] This is discussed in Menger 1951.

[18] Sen 1970, p. 47, which introduced acyclicity in this context.

[19] Sen 1970, theorem 4*1, p. 52 (see also p. 28); lemma 1*l, p. 16 (see also Sen 1997, p. 14, n. 29).

[20] PM, for example, will not avert this problem even if ties are averted in the $R_{\rho i}$ by requiring them to be antisymmetric—unless, that is, the $R_{\rho i}$ exhibit 'value restriction,' a condition explained in (B) below. See Sen 1970, pp. 48–49, and theorem 10*6, p. 184.

[21] Gibbard 1969 as cited in Sen 1970, pp. 49–50, 76, and Sen 1997, pp. 15, 166–167. The proof is recounted in Sen 1986, p. 1085.

[22] These characterizations are given by Kelly 1978, p. 41.

[23] This is shown in Blair and Pollak 1982.

[24] On developing domain restrictions for schemes other than PM, see Sen 1997, p. 12, n. 26.

[25] This result in Sen 1966 unified results concerning 'single-peaked' preferences and other advantageous domain restrictions.

[26] Sen 1970, theorem 10*1, pp. 171–172, 177.

[27] Sen 1970, theorems 10*4, 10*7 and definition 10*9, pp. 174, 179, 183, 185.

[28] Comparing 'conventional' plurality voting to Borda's 'new' scheme, Condorcet wrote that the latter 'confuses votes comparing Peter and Paul with those comparing either Peter or Paul to James and uses them to judge the relative merits of Peter and Paul . . . . [I]t relies on irrelevant factors to form its judgments. . . . . The conventional method is flawed because it ignores elements which should be taken into account and the new one because it takes into account elements which should be ignored' (Condorcet 1788 in McLean and Urken 1995, p. 126). This stance (though not, as noted, the CWC itself) is an early form of [iii].

Condorcet also complained that in single-vacancy elections, the Borda count sometimes flouts the favorite of a majority. But in Michael Dummett's view, this shows to the scheme's advantage in respect of its effect of 'reducing the chances of divisive candidates' for representative office: 'Even a candidate who is the first choice of an absolute majority of the voters will fail to obtain the highest Borda count if he is sufficiently divisive' (Dummett 1998).

[29] This is proved in Gibbard 1973.

[30] In this scheme originated by Thomas Hare and used of late in Northern Ireland and by the Eastern Division of the American Philosophical Association, voters order alternatives by ballot. Any alternative atop a majority of ballots is declared the winner. Failing such a majority favorite, that alternative atop the fewest ballots is deemed stricken from all ballots, which ballots are then reexamined for a majority favorite, failing which the alternative then

atop the fewest ballots is deemed stricken and the ballots are reexamined, and so on until some alternative tops a majority of ballots as trimmed. This scheme too fails the CWC and [iii].

[31] References to studies establishing this correlation between variety of preferences and cycles are given in Plott 1993, p. 252, and Inman 1987, p. 710.

[32] The scheme described in this and the following two paragraphs is set forth in Zwicker 1991.

[33] 'The adjective "weak" refers to the fact that the ordering does not exclude indifference, i.e., axioms I and II do not exclude the possibility that for some distinct x and y, xRy and yRx' (Arrow 1963, p. 13).

[34] In virtue of which Russell understood an ordering to be 'aliorelative,' here using C. S. Pierce's term for irreflexivity (Russell 1920, pp. 31–34). Asymmetry is nowadays demanded only of 'strict' orderings. Arrow acknowledged the prevalent understanding that every ordering is antisymmetric (Arrow 1963, p. 14, n. 8). Elsewhere (in discussing VR, *ibid.*, p. 77), Arrow labeled as a 'strong ordering' a transitive, connected, irreflexive relation, a relation identical with an asymmetric 'preference relation' $P \subset R \in \mathrm{T}$ distinguished by him from a 'weak ordering' $R \in \mathrm{T}$.

[35] The following correspondence obtains between the economist's terms, in italics, and the mathematician's:

<table>
<tr><td>preordering</td><td colspan="2">weak ordering or ordering</td><td>strong ordering</td></tr>
<tr><td rowspan="2">partial preordering</td><td colspan="2">connected preordering</td><td rowspan="2">strict linear ordering</td></tr>
<tr><td>weak total tiering</td><td>weak linear ordering</td></tr>
</table>

[36] E.g., Sen 1970, pp. 9, 28, 41; Kelly 1978, pp. 6, 8, 39; and Dummett 1984, pp. 31, 36, 50.

[37] In this manner Arrow 1967, pp. 216–217, expresses the requirement that indifference be transitive.

[38] The latter result is presented in Fishburn and Gehrlein 1980.

[39] Arrow 1963, consequence 3, p. 50, invoked in the proof at p. 51.

[40] Sen adds that were it not for the envisoned collapse, 'we would have been able to strengthen Arrow's theorem immediately' (Sen 1970, p. 49).

[41] One could also replace [i][2] with '$\mathrm{T} = \mathrm{B} = \Psi$,' producing a theorem about aggregation of positionings into a positioning, but that would be weaker than the above theorem for [2a] and [2c] separately, the T specified in which partition the original T.

[42] This rationale accompanies the introduction of axiom I (Arrow 1963, p. 13).

[43] Sen 1970, pp. 2–3 (on suitable conditions for binary relations) and 14 (on definition 1*8 of a choice function).

[44] Berkeley 1710, Introduction, ¶ 3.

[45] Should a given $C(U)$ contain more than one member, then for distinct x, $y \in C(U)$, $xRy \cdot yRx$, which is to say that any rationalization R is nonsymmetric and hence either nonreflexive or reflexive. For some C, a rationalization will be transitive, irreflexive, and asymmetric.

[46] Cantor 1895 in Jourdain 1952, p. 132; Birkhoff 1948, pp. 1, 10, 31–32; Debreu 1954; Debreu 1959, pp. 56–59, 73.

[47] Sen 1970, theorem 8*2 on 'social welfare functionals,' p. 129.

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